Expected Shortfall and Beyond

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Abstract

Financial institutions have to allocate so-called *economic capital* in order to guarantee solvency to their clients and counterparties. Mathematically speaking, any methodology of allocating capital is a *risk measure*, i.e. a function mapping random variables to the real numbers. Nowadays *value-at-risk*, which is defined as a fixed level quantile of the random variable under consideration, is the most popular risk measure. Unfortunately, it fails to reward diversification, as it is not *subadditive*.

In the search for a suitable alternative to value-at-risk, *Expected Shortfall* (or *conditional value-at-risk* or *tail value-at-risk*) has been characterized as the smallest *coherent* and *law invariant* risk measure to dominate value-at-risk. We discuss these and some other properties of Expected Shortfall as well as its generalization to a class of coherent risk measures which can incorporate higher moment effects. Moreover, we suggest a general method on how to attribute Expected Shortfall *risk contributions* to portfolio components.

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1 Introduction

At the latest in 1999, when the article Artzner et al. (1999) appeared, it became clear that value-at-risk (see Definition 2.1 below) cannot be considered a sound methodology for allocating economic capital in financial institutions. However, even if in Artzner et al. (1999) recommendations were given for the properties sound risk measures should satisfy, only recently

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Expected Shortfall (Definition 3.1 below) was suggested as practicable and sound alternative to value-at-risk. Nevertheless, there are still a lot of useful properties of Expected Shortfall and its generalizations which cannot be found in printed sources so far.

With the paper at hand, we try to make up for this omission. We will recapitulate in section 2 what makes value-at-risk a seductive measure of risk and what are the main criticisms against it. In particular, we will see a new example (Example 2.4) for its lacking subadditivity and give a new interpretation (Remark 2.6) why this is an important point.

We will then introduce in section 3 Expected Shortfall as a convincing alternative to value-at-risk. We will summarize some of its more important properties. These properties are shared by all the representatives of the class of spectral risk measures that were introduced in Acerbi (2002) (cf. Remark 3.7 below). Generalizing a result from Kusuoka (2001), we show that all the elements of this class can be represented as certain averages of values-at-risk at different levels (Theorem 3.6). This representation allows the easy creation of risk measures which enjoy the useful properties of Expected Shortfall and incorporate other desirable features like moment effects.

When a risk measure for a portfolio has been chosen the question arises how to attribute risk contributions to subportfolios. This is of interest for a risk diagnostics of the portfolio (see Litterman, 1996) or for performance analysis. In section 4, we present a suggestion of how to do this in case of spectral risk measures (Definition 4.1 and Proposition 4.2). Finally, we show for the Expected Shortfall that these contributions can be interpreted as conditional expectations given a worst case scenario (Proposition 4.7).

2 Value-at-Risk: lacking subadditivity

Consider a random variable X which might be seen as the random profit and loss of an investment by a fixed time horizon. Positive values of X are regarded as profits, negative values as losses. The value-at-risk (VaR) of X at level α is the absolute value of the worst loss not to be exceeded with a probability of at least α. The following couple of definitions gives a formal description of this quantity.

Definition 2.1 (Quantile, value-at-risk) *Let* $\alpha \in (0, 1]$ *be fixed and* X *be a real random variable on a probability space* $(\Omega, \mathcal{F}, \mathrm{P})$*. Define* $\inf \emptyset = \infty$*. We then call*

$$q_\alpha(X) \;=\; \inf\big\{x \in \mathbb{R} : \mathrm{P}[X \le x] \ge \alpha\big\} \tag{2.1a}$$

the α-quantile *of* X*. We call*

$$\mathrm{VaR}_\alpha(X) \;=\; q_\alpha(-X) \tag{2.1b}$$

the value-at-risk (VaR) *at (confidence) level* α *of* X*.*

Usually, values of α close to 1 are of interest. Since by definition $\mathrm{P}\big[X + \mathrm{VaR}_\alpha(X) \geq 0\big] \geq \alpha$, $\mathrm{VaR}_\alpha(X)$ can be interpreted as the minimal amount of capital to be put back by the investor in order to preserve her solvency with a probability of least α.

Below, we will compare VaR to other methods for attributing capital to random variables (sometimes in insurance contexts also called risks). A positive capital attribution means that the risk under consideration *requires* capital whereas a negative capital attribution indicates that capital may be released. From an economical point of view, it makes sense to allow for risks which require a positively infinite amount of capital. A risk with capital requirement ∞ must not be accepted by the investor. The interpretation of a risk with capital requirement $-\infty$ is much less straightforward. Would this imply that such a risk can serve as a collateral for any risk with finite capital requirement? However, this case does not appear very likely and is therefore excluded from the following definition of risk measures.

Definition 2.2 (Risk measure) *Let* $(\Omega, \mathcal{F}, \mathrm{P})$ *be a probability space and* V *be a non-empty set of* $\mathcal{F}$*-measurable real-valued random variables. Then any mapping* $\rho: V \to \mathbb{R} \cup \{\infty\}$ *is called a* risk measure*.*

VaR, as a risk measure in the sense of Definition 2.2, enjoys most of the properties that are considered useful in the literature (Artzner et al., 1999; Kusuoka, 2001).

Proposition 2.3 (Properties of value-at-risk) *Let* $\alpha \in (0, 1]$ *be fixed and* $(\Omega, \mathcal{F}, \mathrm{P})$ *be a probability space. Consider the risk measure* ρ *on the set* V *of all the* $\mathcal{F}$*-measurable real-valued random variables which is given by*

$$\rho(X) \;=\; \mathrm{VaR}_\alpha(X), \quad X \in V. \tag{2.2}$$

Then ρ *has the following properties:*

(1) Monotonicity: $X, Y \in V$*,* $X \leq Y \;\Rightarrow \rho(X) \geq \rho(Y)$*.*

(2) Positive homogeneity: $X \in V, h > 0, h\,X \in V \;\Rightarrow \rho(h\,X) = h\,\rho(X)$*.*

(3) Translation invariance: $X \in V, a \in \mathbb{R}, X + a \in V \Rightarrow \rho(X + a) = \rho(X) - a$*.*

(4) Law invariance: $X, Y \in V$*,* $\mathrm{P}[X \leq t] = \mathrm{P}[Y \leq t]$ *for all* $t \in \mathbb{R}$
$\Rightarrow \rho(X) = \rho(Y)$*.*

(5) Comonotonic additivity: f, g *non-decreasing,* Z *real random variable on* $(\Omega, \mathcal{F}, \mathrm{P})$ *such that* $f \circ Z, g \circ Z \in V$
$\Rightarrow \rho(f \circ Z + g \circ Z) = \rho(f \circ Z) + \rho(g \circ Z)$*.*

Proof. (1) until (4) are obvious. For (5), see e.g. Denneberg (1994). $\square$

Note that VaR_α is law invariant in a very strong sense: the distributions of X and Y need not be identical in order to imply $\mathrm{VaR}_\alpha(X) = \mathrm{VaR}_\alpha(Y)$. A certain local identity of the distributions suffices for this implication. In particular, random variables X with light tail probabilities and Y with heavy tail probabilities (see e.g. Embrechts et al., 1997) may have the same VaR_α. This point is one main criticism against VaR as a risk measure.

One important property is missing in the enumeration of Proposition 2.3: the *subadditivity*, i.e.

$$X, Y \in V, X + Y \in V \;\Rightarrow\; \rho(X+Y) \le \rho(X) + \rho(Y). \tag{2.3}$$

It is well-known that VaR is *not* in general subadditive. Here we present a counterexample with continuous and even independent random variables.

Example 2.4 (Lacking subadditivity of VaR) *Let* X_1, X_2 *Pareto distributed with values in* $(-\infty, 1)$ *and independent. The joint distribution of* (X, Y) *is specified by*

$$\mathrm{P}[X_1 \le x_1, X_2 \le x_2] \;=\; (2 - x_1)^{-1}(2 - x_2)^{-1}, \quad x_1, x_2 < 1. \tag{2.4a}$$

This implies

$$\begin{aligned} \mathrm{VaR}_\alpha(X_i) &= (1-\alpha)^{-1} - 2, \qquad & i = 1, 2, \\ \mathrm{P}[X_1 + X_2 \le x] &= \frac{2}{4-x} + \frac{2\,\log(3-x)}{(4-x)^2}, \qquad & x < 2. \end{aligned} \tag{2.4b}$$

By (2.4b), we have $\mathrm{VaR}_\alpha(X_1) + \mathrm{VaR}_\alpha(X_2) < \mathrm{VaR}_\alpha(X_1 + X_2)$ *for all* $\alpha \in (0, 1)$ *because*

$$\begin{aligned} \mathrm{P}\big[-(X_1+X_2) \le 2\,\mathrm{VaR}_\alpha(X_1)\big] &= \alpha - \tfrac{1-\alpha}{2}\,\log\tfrac{1+\alpha}{1-\alpha} \\ &< \alpha. \end{aligned} \tag{2.4c}$$

In particular, for $\alpha = 0,99$ *we have*

$$\mathrm{VaR}_\alpha(X_1) = \mathrm{VaR}_\alpha(X_2) = 98, \quad \mathrm{VaR}_\alpha(X_1 + X_2) \approx 203,2.$$

The lacking subadditivity of VaR is criticized because under certain circumstances it might be an incentive to split up a large firm into two smaller firms. Another interpretation (Remark 2.6) follows from the following result.

Proposition 2.5 *Let* X, Y *be real, linearly independent random variables and* ρ *be a real-valued risk measure on the positive cone* C *spanned by* X *and* Y, *i.e.* $C = \{u\,X + v\,Y : u, v > 0\}$. *Assume that* ρ *is positively homogeneous*

in the sense of Proposition 2.3 (2) and that the function $\rho(u,v) = \rho(u\,X + v\,Y)$, $u, v > 0$ *is differentiable in* (u,v). *Then we have*

$$\rho(U_1 + U_2) \;\le\; \rho(U_1) + \rho(U_2), \quad U_1, U_2 \in C, \tag{2.5a}$$

if and only if

$$\rho_{U_1}(U_1 + U_2) \le \rho(U_1),\;\; \rho_{U_2}(U_1 + U_2) \le \rho(U_2), \qquad U_1, U_2 \in C, \tag{2.5b}$$

with

$$\rho_{U_i}(U_1 + U_2) = u_i\,\frac{\partial \rho}{\partial u}\big(u_1 + u_2, v_1 + v_2\big) + v_i\,\frac{\partial \rho}{\partial v}\big(u_1 + u_2, v_1 + v_2\big)$$

when $U_i = u_i\,X + v_i\,Y$, $i = 1, 2$ *(note that by linear independence this representation is unique).*

Remark 2.6 *By Euler's relation (see (2.6d)), the terms* $\rho_{U_1}(U_1 + U_2)$ *and* $\rho_{U_2}(U_1 + U_2)$ *from (2.5b) sum up to* $\rho(U_1 + U_2)$. *Hence it appears quite natural to regard them as the risk (or capital) contributions of* U_1 *and* U_2 *respectively to the total capital* $\rho(U_1 + U_2)$ *which is required by* $U_1 + U_2$. *Indeed, it can be argued that there is no other way to arrive at a reasonable notion of capital contribution than by partial derivatives (cf. Denault, 2001; Tasche, 1999). Moreover,* VaR *and the risk measure* ES *to be defined below (Definition 3.2) satisfy the conditions of Proposition 2.5 under quite general assumptions on the joint distribution of* (X, Y) *(cf. Tasche, 2000).*

With this interpretation of $\rho_{U_i}(U_1 + U_2)$, $i = 1, 2$, *the meaning of (2.5b) is as follows: the manager who is responsible for subportfolio* U_1 *will never be damaged by diversification in the portfolio of the firm because her capital contribution will never be greater than the capital requirement in the case of* U_1 *considered as a stand-alone portfolio.*

Proof of Proposition 2.5. We show first that (2.5a) implies $\rho_{U_1}(U_1+U_2) \le \rho(U_1)$. Fix $U_i = u_i\,X + v_i\,Y$, $i = 1, 2$, and note that $\rho(U_1) = \rho(u_1, v_1)$ and $\rho(U_1 + U_2) = \rho(u_1 + u_2, v_1 + v_2)$. Define the function $f : (-1, \infty) \to \mathbb{R}$ by

$$\begin{aligned} f(t) = \rho(u_1 + u_2, v_1 + v_2) &+ t\,\rho(u_1, v_1) \\ &- \rho\big((1+t)\,u_1 + u_2, (1+t)\,v_1 + v_2\big). \end{aligned} \tag{2.6a}$$

Then

$$\begin{aligned} f'(t) = \rho(u_1, v_1) &- u_1\,\frac{\partial \rho}{\partial u}\big((1+t)\,u_1 + u_2, (1+t)\,v_1 + v_2\big) \\ &- v_1\,\frac{\partial \rho}{\partial v}\big((1+t)\,u_1 + u_2, (1+t)\,v_1 + v_2\big) \end{aligned} \tag{2.6b}$$

and in particular

$$
\begin{aligned}
f(0) &= 0, \\
f'(0) &= \rho(u_1, v_1) - u_i \frac{\partial \rho}{\partial u}\big(u_1 + u_2, v_1 + v_2\big) + v_i \frac{\partial \rho}{\partial v}\big(u_1 + u_2, v_1 + v_2\big) \\
&= \rho(U_1) - \rho_{U_1}(U_1 + U_2).
\end{aligned}
\tag{2.6c}
$$

(2.5a) implies for $t > 0$ that $f(t) \geq 0$. But, by (2.6c), this is a contradiction to the assumption $\rho(U_1) - \rho_{U_1}(U_1 + U_2) = f'(0) < 0$. This implies (2.5b).

Let us now consider the proof of the implication (2.5b) $\Rightarrow$ (2.5a). This is easy since by Euler's relation and (2.5b)

$$
\begin{aligned}
\rho(U_1 + U_2) &= \rho(u_1 + u_2, v_1 + v_2) \\
&= (u_1 + u_2)\,\frac{\partial \rho}{\partial u}\big(u_1 + u_2, v_1 + v_2\big) \\
&\qquad - \;(v_1 + v_2)\,\frac{\partial \rho}{\partial v}\big(u_1 + u_2, v_1 + v_2\big) \\
&\leq \rho(U_1) + \rho(U_2).
\end{aligned}
\tag{2.6d}
$$

This completes the proof of Proposition 2.5. □

3 Spectral risk measures

The weak points of VaR as a risk measure are well-known for some time (cf. Artzner et al., 1999). Nowadays, there is a certain consensus on the properties a reasonable risk measure should satisfy (Artzner et al., 1997, 1999; Delbaen, 1998; but see also Föllmer and Schied, 2002, for a relaxation): it should be coherent in the sense of the following definition.

Definition 3.1 (Coherent risk measure) *A risk measure $\rho : V \to \mathbb{R} \cup \{\infty\}$ in the sense of Definition 2.2 is called* coherent *if it is monotonous, positively homogeneous, translation invariant, and subadditive (see Proposition 2.3 (1), (2), (3), and Eq. (2.3)).*

In order to preserve the desirable connection between the level of VaR_α and the investor's probability of solvency, it would be nice to have a smallest coherent risk measure to dominate VaR_α. As was shown in Delbaen (1998), such a smallest coherent majorant to VaR_α does not exist. Nevertheless, in Delbaen (1998) was also shown there is a smallest coherent and *law invariant* (see Proposition 2.3 (4)) risk measure[1]that dominates VaR_α. The

[1] The term "law invariance" was introduced in Kusuoka (2001). A rough interpretation of law invariance might be "can be estimated from statistical observations only". Anyway,

representation of this measure in Delbaen (1998) was not explicit in the general case. However, it became clear that for continuous random variables X, it coincides with $\mathrm{E}\big[-X \mid -X \le \mathrm{VaR}_\alpha(X)\big]$, the so-called *tail value-at-risk*. Note that tail value-at-risk, in general, is not subadditive (see e.g. Acerbi and Tasche, 2002).

Denote – as usual – by $\mathbf{1}_A = \mathbf{1}_A(a)$ the *indicator function* of the set A, i.e. $\mathbf{1}_A(a) = 0$ if $a \notin A$ and $\mathbf{1}_A(a) = 1$ if $a \in A$.

Definition 3.2 *Let* $\alpha \in (0, 1)$ *be fixed and* X *be a real random variable on a probability space* $(\Omega, \mathcal{F}, \mathrm{P})$ *with* $\mathrm{E}\big[\max(0, -X)\big] < \infty$. *Define* $q_\alpha(-X)$ *as in Definition 2.1. We then call*

$$\mathrm{ES}_\alpha(X) = -(1-\alpha)^{-1}\Big(\mathrm{E}\big[X\,\mathbf{1}_{\{-X \ge q_\alpha(-X)\}}\big] + q_\alpha(-X)\Big\{\alpha - \mathrm{P}\big[-X < q_\alpha(-X)\big]\Big\}\Big) \qquad (3.1)$$

Expected Shortfall (ES) *at level* α *of* X.

It turned out (Kusuoka, 2001; Acerbi and Tasche, 2002) that ES from Definition 3.2 is just the smallest coherent and law invariant majorant of VaR_α which had been already mentioned in Delbaen (1998). The term ES stems from Acerbi et al. (2001) where a further proof of the coherence of ES was given. Independently, ES was introduced in Rockafellar and Uryasev (2001) under the notion *Conditional value-at-risk (CVaR)*. The properties of ES are discussed in detail in Acerbi and Tasche (2002) and Rockafellar and Uryasev (2001).

The following result (Acerbi and Tasche, 2002; Pflug, 2000) is important for the calculation of VaR and ES, and, by the way, enlightens the relationship between the notion of ES and the *quantile regression* which was introduced in Koenker and Bassett (1978). ES is just the optimal value in an optimization problem where $-\mathrm{VaR}$ is the optimizing argument.

Proposition 3.3 *For* ES_α *as given in Definition 3.2 and* $q_\alpha, q_{1-\alpha}$ *as given in Definition 2.1, we have*

$$\mathrm{ES}_\alpha(X) = \min_{s\in\mathbb{R}} -(1-\alpha)^{-1}\Big(\mathrm{E}\big[X\,\mathbf{1}_{\{-X\ge s\}}\big] + s\Big\{\alpha - \mathrm{P}\big[-X < s\big]\Big\}\Big) \qquad (3.2a)$$

and

$$\big[q_\alpha(-X), -q_{1-\alpha}(X)\big] = \arg\min_{s\in\mathbb{R}} -(1-\alpha)^{-1}\Big(\mathrm{E}\big[X\,\mathbf{1}_{\{-X\ge s\}}\big] + s\Big\{\alpha - \mathrm{P}\big[-X < s\big]\Big\}\Big), \qquad (3.2b)$$

as VaR is law invariant it seems natural to look for its smallest coherent *and* law invariant majorant. See eq. (1) in Acerbi (2002) for an example of a risk measure which is not law invariant in sense of Proposition 2.3 (4).

whenever X is a real random variable with $\mathrm{E}\big[\max(0, -X)\big] < \infty$.

Proof. Proposition 4.2 in Acerbi and Tasche (2002). □

Note that the interval in (3.2b) is never empty and that $\mathrm{VaR}_\alpha(X) = q_\alpha(-X)$ by definition. Let us now have a look on another useful representation of ES.

Proposition 3.4 *For* ES_α *as given in Definition 3.2 and* VaR_α *as given in Definition 2.1, we have*

$$\mathrm{ES}_\alpha(X) = (1-\alpha)^{-1} \int_\alpha^1 \mathrm{VaR}_u(X)\, du, \tag{3.3}$$

whenever X is a real random variable with $\mathrm{E}\big[\max(0, -X)\big] < \infty$.

Proof. Proposition 3.2 in Acerbi and Tasche (2002). □

In combination with Proposition 2.3, Proposition 3.4 implies that ES is a law invariant and comonotonic additive risk measure. The comonotonic additivity of a risk measure becomes particularly interesting when it occurs at the same time as subadditivity.

Remark 3.5 *Fix $\alpha \in (0, 1)$ and consider integrable random variables X and Y. Assume that we do not know the joint distribution of X and Y. Then, from subadditivity, we see that $\mathrm{ES}_\alpha(X) + \mathrm{ES}_\alpha(Y)$ is an upper bound for the risk of $X + Y$ when risk is measured by* ES. *By comonotonic additivity, we know additionally that this upper bound is sharp in the sense that it occurs in the case of comonotonic X and Y (i.e. $X = f \circ Z$ and $Y = g \circ Z$ for some random variable Z and non-decreasing functions f and g).*

Compare this to the situation when VaR_α is used as risk measure. Then there is no easy general upper bound for the risk of $X + Y$, and finding the joint distribution of X and Y which yields the maximum value for $\mathrm{VaR}_\alpha(X+Y)$ is a non-trivial task (Embrechts et al., 2001; Luciano and Marena, 2001).

Note that there are coherent and law invariant risk measures which are not comonotonic additive (e.g. the standard semi-deviation, see Fischer, 2001).

It might have become clear from the above considerations that the class of coherent, law invariant and comonotonic additive risk measures is of particular interest. In Kusuoka (2001), a complete characterization of this class was accomplished, under the additional assumption that the risk measures under consideration satisfy the so-called *Fatou property*. We show that this assumption is dispensable.

Theorem 3.6 *Let ρ be a risk measure on the space V of the bounded random variables in the probability space $(\Omega, \mathcal{F}, \mathrm{P})$. Assume that $(\Omega, \mathcal{F}, \mathrm{P})$ is standard and non-atomic (i.e. there exists a random variable which is uniformly*

distributed on $(0,1)$*). Then* ρ *is a coherent, law invariant and comonotonic additive (see Definition 3.1 and Proposition 2.3 (4), (5)) risk measure if and only if*

$$\rho(X) = p \int_0^1 \mathrm{VaR}_u(X)\, F(du) + (1-p)\, \mathrm{VaR}_1(X), \quad X \in V, \tag{3.4}$$

where $p \in [0,1]$ *and* F *is a continuous convex distribution function which is concentrated on* $[0,1]$*.*

Remark 3.7

(i) Choose $p = 1$ *and* $F(u) = \max\bigl(0, \frac{u-\alpha}{1-\alpha}\bigr)$ *in order to obtain* ES_α *from (3.4).*

(ii) Note that any continuous and convex distribution function F *on* $[0,1]$ *is absolutely continuous, i.e. can be written as* $F(u) = \int_0^u f(t)\,dt$ *where* f *is its density with respect to Lebesgue measure. Thus Theorem 3.6 states that the class of* spectral risk measures *which was introduced in Acerbi (2002) is just the class of coherent, law invariant and comonotonic additive risk measures.*

(iii) Formulas like (3.4) can be traced back a long time in the actuarial literature (cf. Wang, 1996, and the references therein).

Proof of Theorem 3.6. Let us first regard the case where a risk measure ρ as in (3.4) is given. Law invariance of ρ is then clear since (3.4) is based on quantiles of X. If $p = 0$ then ρ is just the essential supremum of X. It is then obvious that ρ is coherent and comonotonic additive.

Assume now $p > 0$. Construct a function $F_0 : [0,1] \to [0,1]$ by setting

$$F_0(u) = \begin{cases} p\, F(u), & 0 \le u < 1 \\ 1, & u = 1. \end{cases} \tag{3.5}$$

Observe that F_0 is again convex and non-decreasing but may fail to be continuous in 1. Nevertheless, it is easy to show that (3.4) is equivalent to

$$\rho(X) = -\int X\, dF_0 \circ \mathrm{P}, \quad X \in V, \tag{3.6}$$

where $\int X\, dF_0 \circ \mathrm{P}$ denotes the non-additive integral with respect to the *distorted probability* $F_0 \circ \mathrm{P}$ in the sense of Denneberg (1994). Coherence and comonotonic additivity of ρ are now just conclusions from the general theory of non-additive integration.

Next, we show that any coherent, law invariant and comonotonic additive risk measure ρ can be represented as in (3.4). As a first step, we conclude

from the results in Schmeidler (1986, Schmeidler's theorem) that ρ can be written as

$$\rho(X) = -\int X\, d\nu, \quad X \in V, \tag{3.7}$$

where $\int X\, d\nu$ denotes again a non-additive integral in the sense of Denneberg (1994). ν is a monotonous (i.e. $A, B \in \mathcal{F}, A \subset B \Rightarrow \nu(A) \le \nu(B)$) and super-modular (i.e. $A, B \in \mathcal{F} \Rightarrow \nu(A) + \nu(B) \le \nu(A \cup B) + \nu(A \cap B)$) set function (i.e. $\nu(\emptyset) = 0$) on $(\Omega, \mathcal{F})$ with $\nu(\Omega) = 1$. We define a function $F_0 : [0,1] \to [0,1]$ by

$$F_0(u) \;=\; \nu(A) \tag{3.8a}$$

for any $A \in \mathcal{F}$ with $\mathrm{P}[A] = u$. Since $(\Omega, \mathcal{F}, \mathrm{P})$ is standard and non-atomic, for every $u \in [0,1]$ there is at least one $A \in \mathcal{F}$ with $\mathrm{P}[A] = u$. The law invariance of ρ implies that F_0 is with (3.8a) is well-defined. The monotonicity of ν implies that F_0 is non-decreasing. Moreover, from (3.8a) also follows

$$\nu = F_0 \circ \mathrm{P}. \tag{3.8b}$$

Again, since $(\Omega, \mathcal{F}, \mathrm{P})$ is standard and non-atomic, for any $u_1, u_2, u_3, u_4 \in [0,1]$ with $0 \le u_1 < u_4 \le 1$, $u_2, u_3 \in [u_1, u_4]$ and $u_2 - u_1 = u_4 - u_3$ there are events $A, B \in \mathcal{F}$ such that we have

$$\mathrm{P}[A] = u_3,\ \mathrm{P}[B] = u_2,\ \mathrm{P}[A \cap B] = u_1,\ \mathrm{P}[A \cup B] = u_4. \tag{3.9}$$

The super-modularity of ν and (3.9) imply

$$F_0(u_4) + F_0(u_1) \ge F_0(u_2) + F_0(u_3) \tag{3.10}$$

for all u_1, u_2, u_3, u_4 as above. With $u_2 = u_3$, (3.10) yields for any $0 \le u < v \le 1$ that

$$F_0\big(\frac{u+v}{2}\big) \le \frac{1}{2}\, F_0(u) + \frac{1}{2}\, F_0(v). \tag{3.11a}$$

Of course, from (3.11a) we obtain

$$F_0\big(\alpha\, u + (1-\alpha)\, v\big) \le \alpha\, F_0(u) + (1-\alpha)\, F_0(v) \tag{3.11b}$$

for every $\alpha \in \big\{\frac{k}{2^n} : n \ge 1, k = 0, 1, \ldots, 2^n\big\}$. Since F_0 is non-decreasing, limits from the right exist in $\alpha\, u + (1-\alpha)\, v$ for every $\alpha \in (0,1)$. Hence, by passing to the limits in (3.11b) we can conclude that (3.11b) holds for every $\alpha \in [0,1]$, i.e. F_0 is convex. Observe that a function $F_0 : [0,1] \to [0,1]$ with $F_0(0) = 0$ and $F_0(1) = 1$ is necessarily continuous on $[0,1)$ if it is non-decreasing and convex. Furthermore, F_0 can be constant at most on an interval $[0, \epsilon)$. On $[\epsilon, 1]$ it will then be strictly increasing.

So far, we know that ρ can be represented by (3.6) where $F_0 : [0,1] \to [0,1]$ is non-decreasing and convex as described above. Now, applying the definition and some other properties of non-additive integrals yields representation (3.4) where p is given by $p = \sup_{u \in [0,1)} F_0(u)$ and F and F_0 are related by (3.5). $\square$

Remark 3.8 (Incorporating moment effects in ES)
Representation (3.6) allows in some cases a helpful interpretation of what happens when $\rho(X)$ is calculated. Fix any positive integer n. Recall from Remark 3.7 (i) the function F which generates ES when used in (3.4). Define

$$M_n(u) \;=\; F(1 - \sqrt[n]{1-u}), \quad u \in [0,1]. \tag{3.12}$$

If X is a real random variable, interpreted as the profit and loss of a financial asset, consider independent and identically distributed $Y, Y_1, \ldots, Y_n$ with $\mathrm{P}[-Y \le t] = 1 - \sqrt[n]{\mathrm{P}[-X > t]}$. *Then*

$$\begin{aligned} \mathrm{P}[-X \le t] &= \mathrm{P}\big[\min(-Y_1, \ldots, -Y_n) \le t\big], \\ F\big(\mathrm{P}[Y > t]\big) &= M_n\big(\mathrm{P}[X > t]\big), \end{aligned} \tag{3.13}$$

and hence

$$\int Y\, d\, F \circ \mathrm{P} = \int X\, d\, M_n \circ \mathrm{P}. \tag{3.14}$$

By Theorem 3.6, hence $\mathrm{ES}^{(n)}_\alpha(X) = \mathrm{ES}_\alpha(Y)$ *is a spectral risk measure. Note that (cf. Delbaen, 1998)*

$$\mathrm{E}[\max(0, -Y)] < \infty \;\Rightarrow\; \mathrm{E}\big[(\max(0,-X))^n\big] < \infty \tag{3.15a}$$

and for any $\epsilon > 0$

$$\mathrm{E}\big[(\max(0,-X))^{n+\epsilon}\big] < \infty \;\Rightarrow\; \mathrm{E}[\max(0,-Y)] < \infty. \tag{3.15b}$$

(3.15a) and (3.15b) show that $\mathrm{ES}^{(n)}_\alpha(X)$ *is sensible to the n-th moment of X. By (3.13) it may be interpreted as the Expected Shortfall of a random variable Y which is generated from X by a pessimistic manipulation since the loss variable $-X$ has the same distribution as the minimum of n independent copies of the loss $-Y$.*

4 Risk contributions

In this section we study the following problem: Given random variables $X_1, \ldots, X_d$ (e.g. profits and losses of the different business lines in a financial institution), portfolio weights $u_1, \ldots, u_d$, and a risk measure ρ, we want to know how much $u_i\, X_i$ contributes to the total risk $\rho\big(\sum_{i=1}^d u_i\, X_i\big)$ of the portfolio. With $u = (u_1, \ldots, u_d)$ write for short

$$\rho(u) = \rho\Big(\sum_{i=1}^d u_i\, X_i\Big). \tag{4.1}$$

Denault (2001) and Tasche (1999) (with different reasonings) argued that $u_i \frac{\partial \rho}{\partial u_i}(u)$ is the appropriate definition for the *risk contribution* of $u_i\, X_i$ in the case when $\rho(u)$ is partially differentiable with respect to the components of u.

The question of how to compute the partial derivatives in case $\rho = \mathrm{VaR}_\alpha$ was independently tackled by several authors (Gouriéroux et al., 2000; Hallerbach, 1999; Lemus, 1999; Tasche, 1999). They observed that under certain smoothness assumptions on the joint distribution of $(X_1, \ldots, X_d)$

$$\frac{\partial \mathrm{VaR}_\alpha}{\partial u_i}\Big(\sum_{j=1}^d u_j\, X_j\Big) = -\,\mathrm{E}\Big[X_i \,\Big|\, -\sum_{j=1}^d u_j\, X_j = \mathrm{VaR}_\alpha\big(\sum_{j=1}^d u_j\, X_j\big)\Big]. \tag{4.2a}$$

Note that in case of integrable components $X_1, \ldots, X_d$ the right-hand side of (4.2a) will always be defined. By the factorization lemma there are functions $\phi_i = \phi_i(u; z)$, $i = 1, \ldots, d$, such that almost surely

$$\mathrm{E}\Big[X_i \,\Big|\, \sum_{j=1}^d u_j\, X_j = z\Big] = \phi_i(u; z). \tag{4.2b}$$

By inspection of (3.4), (4.2a) and (4.2b) suggest the following definition for risk contributions in case of a spectral risk measure in the sense of Remark 3.7 (ii).

Definition 4.1 (Risk contributions of spectral risk measures) *Let $X_1, \ldots, X_d$ be integrable random variables on the same probability space, $u = (u_1, \ldots, u_d)$ their portfolio weight vector, and ρ be a spectral risk measure with representation (3.4). Define functions $\phi_i = \phi_i(u; z)$, $i = 1, \ldots, d$, by (4.2b) and write $\mathrm{VaR}_\alpha(u)$ for $\mathrm{VaR}_\alpha\big(\sum_{j=1}^d u_j\, X_j\big)$. Then, if all the involved integrals exist, the quantity*

$$\rho_i(u) = -p \int_0^1 \phi_i\big(u; -\mathrm{VaR}_\alpha(u)\big)\, F(d\alpha) - (1-p)\, \phi_i\big(u; -\mathrm{VaR}_1(u)\big) \tag{4.3}$$

is called marginal impact *of X_i on the total risk $\rho(u) = \rho\big(\sum_{j=1}^d u_j\, X_j\big)$. The quantity $u_i\, \rho_i(u)$ is called* risk contribution *of $u_i\, X_i$ to the total risk.*

By the standard theory of non-additive integration (see Denneberg, 1994) we obtain the following equivalent representation of the risk contributions from Definition 4.1.

Proposition 4.2 *Assume that the random variables $X_1, \ldots, X_d$ from Definition 4.1 are defined on the probability space $(\Omega, \mathcal{F}, \mathrm{P})$. Then the marginal impacts $\rho_i(u)$ in (4.3) can be equivalently written as*

$$\rho_i(u) = -\int \phi_i\big(u; \sum_{j=1}^d u_j\, X_j\big)\, dF_0 \circ \mathrm{P}, \tag{4.4}$$

where F_0 is given by (3.5). As a consequence, for fixed u, the value of $\rho_i(u)$ does not depend on the choice of $\phi_i(u;\cdot)$.

Proof. (4.3) $\iff$ (4.4) can be proved like in the proof of Theorem 3.6. Denote $\sum_{j=1}^d u_j\,X_j$ by $Z(u)$, and let $\phi_i(u;z)$ and $\phi_i^*(u;z)$ be two versions of $\mathrm{E}\big[X_i\,|\,Z(u)=z\big]$. Then, $\phi_i(u;z)$ and $\phi_i^*(u;z)$ are equal for all z but those in a set N with $\mathrm{P}\big[Z(u)\in N\big]=0$. Of course, then we also have $(F_0\circ \mathrm{P})\big[Z(u)\in N\big]=0$. This implies

$$\int \phi_i(u;Z(u))\,dF_0\circ\mathrm{P} = \int \phi_i^*(u;Z(u))\,dF_0\circ\mathrm{P}.$$

Thus, the proof is accomplished. □

Remark 4.3 *(4.4) suggests the following procedure for the estimation of the marginal impacts $\rho_i(u)$ on the spectral risk measure $\rho(u)=\rho\big(\sum_{j=1}^d u_j\,X_j\big)$:*

1. *Estimate the conditional expectations $\phi_i(u;\cdot)$ (see (4.2b); could be done by a kernel estimation).*
2. *Estimate the distribution of $\sum_{j=1}^d u_j\,X_j$ (could be done by a kernel estimation of the density).*
3. *Resample from the distribution of $\sum_{j=1}^d u_j\,X_j$ distorted by F_0, apply $\phi_i(u;\cdot)$ on the sample, and estimate $\rho_i(u)$ with the ordinary sample mean.*

The representations (4.3) and (4.4) of the marginal impacts on spectral risk measures can be significantly simplified in case of the Expected Shortfall ES_α (Definition 3.2). To see this we need the following two results.

Proposition 4.4 *Let X be a real random variable, $f:\mathbb{R}\to[0,\infty)$ a function such that $\mathrm{E}\big[\max(0,-f\circ X)\big]<\infty$ and let $\alpha\in(0,1)$ be a fixed confidence level. Then*

$$\begin{aligned}\int_0^\alpha f\big(q_u(X)\big)\,d\,u \;&=\; \mathrm{E}\big[f\circ X\,\mathbf{1}_{\{X\le q_\alpha(X)\}}\big]\\ &\qquad + f\big(q_\alpha(X)\big)\,\big(\alpha-\mathrm{P}[X\le q_\alpha(X)]\big).\end{aligned}\tag{4.5}$$

Proof. By switching to another probability space if necessary, we can assume that there is a real random variable U that is uniformly distributed on $(0,1)$, i.e. $\mathrm{P}[U\le u]=u$, $u\in(0,1)$. It is well-known that then the random variable $Z=q_U(X)$ has the same distribution as X.

Since $u\mapsto q_u(X)$ is non-decreasing we have

$$\begin{gathered}\{U\le\alpha\}\subset\{Z\le q_\alpha(X)\}\quad\text{and}\\ \{U>\alpha\}\cap\{Z\le q_\alpha(X)\}\subset\{Z=q_\alpha(X)\}.\end{gathered}\tag{4.6}$$

By (4.6) we obtain

$$\begin{aligned}\int_0^\alpha f\big(q_u(X)\big)\,d\,u &= \mathrm{E}[f\circ Z\,\mathbf{1}_{\{U\le\alpha\}}]\\ &= \mathrm{E}[f\circ Z\,\mathbf{1}_{\{Z\le q_\alpha(X)\}}] - \mathrm{E}[f\circ Z\,\mathbf{1}_{\{U>\alpha\}\cap\{Z\le q_\alpha(X)\}}]\\ &= \mathrm{E}[f\circ X\,\mathbf{1}_{\{X\le q_\alpha(X)\}}] + q_\alpha(X)\,\big(\alpha - \mathrm{P}[X\le q_\alpha(X)]\big).\end{aligned} \tag{4.7}$$

Thus, the proof is accomplished. □

Remark 4.5 *Prop. 4.4 generalizes Prop. 3.2 of Acerbi and Tasche (2002). The "$\le$" in (4.5) may be replaced by "$<$".*

Corollary 4.6 *Let X, Y be real random variables such that $\mathrm{E}\big[|Y|\big] < \infty$ and let $\alpha \in (0,1)$ be a fixed confidence level. Then*

$$\begin{aligned}\int_\alpha^1 \mathrm{E}\big[Y\,|\,X = -\mathrm{VaR}_u(X)\big]\,du &= \mathrm{E}\big[Y\,\mathbf{1}_{\{-X\ge q_\alpha(-X)\}}\big]\\ &\quad + \mathrm{E}\big[Y\,|\,-X = q_\alpha(-X)\big]\Big(\mathrm{P}\big[-X < q_\alpha(-X)\big] - \alpha\Big).\end{aligned} \tag{4.8}$$

Moreover, the value of $\int_\alpha^1 \mathrm{E}\big[Y\,|\,X = -\mathrm{VaR}_u(X)\big]\,du$ is the same for any version of the conditional expectation.

Proof. Non-dependence on the particular version of conditional expectation follows from Proposition 4.4. Observe that

$$\begin{aligned}\int_\alpha^1 \mathrm{E}\big[Y\,|\,X = -\mathrm{VaR}_u(X)\big]\,du &= \int_0^1 \mathrm{E}\big[Y\,|\,-X = q_u(-X)\big]\,du\\ &\quad - \int_0^\alpha \mathrm{E}\big[Y\,|\,-X = q_u(-X)\big]\,du\\ &= \mathrm{E}[Y] - \int_0^\alpha \mathrm{E}\big[Y\,|\,-X = q_u(-X)\big]\,du.\end{aligned} \tag{4.9a}$$

Proposition 4.4 and Remark 4.5, applied to $f(x) = \mathrm{E}\big[Y\,|\,-X = x\big]$, yield

$$\begin{aligned}\int_0^\alpha \mathrm{E}\big[Y\,|\,-X = q_u(-X)\big]\,du &= \mathrm{E}\big[Y\,\mathbf{1}_{\{-X< q_\alpha(-X)\}}\big]\\ &\quad + \mathrm{E}\big[Y\,|\,-X = q_\alpha(-X)\big]\Big(\alpha - \mathrm{P}\big[-X < q_\alpha(-X)\big]\Big).\end{aligned} \tag{4.9b}$$

(4.9a) and (4.9b) imply the assertion. □

Recall Definition 4.1 of the marginal impact $\rho_i(u)$ of a component X_i on the total risk $\rho(u)$ of a portfolio $\sum_{j=1}^d u_j\,X_j$ when ρ is a spectral risk measure.

This definition applies to $\rho = \mathrm{ES}_\alpha$ with $p = 1$ and $F(u) = \max\bigl(0, \frac{u-\alpha}{1-\alpha}\bigr)$. In this case, Corollary 4.6 implies the representation (with $Z(u) = \sum_{j=1}^d u_j\, X_j$)

$$\rho_i(u) = -(1-\alpha)^{-1}\Bigl\{\mathrm{E}\bigl[X_i\, \mathbf{1}_{\{-Z(u) \ge q_\alpha(-Z(u))\}}\bigr]$$
$$+\, \mathrm{E}\bigl[X_i \,|\, -Z(u) = q_\alpha(-Z(u))\bigr]\Bigl(\mathrm{P}\bigl[-Z(u) < q_\alpha(-Z(u))\bigr] - \alpha\Bigr)\Bigr\}. \quad (4.10)$$

Returning to (4.8), we will show that its right-hand side times $(1-\alpha)^{-1}$ (and, as a consequence, also $\rho_i(u)$ from(4.10)) can be interpreted as a conditional expectation given that a certain worst case event has occurred. Note first that by the very definition of quantile we have

$$0 \le \mathrm{P}\bigl[-X \le q_\alpha(-X)\bigr] - \alpha \le \mathrm{P}\bigl[-X = q_\alpha(-X)\bigr] \quad (4.11a)$$

and in particular

$$\mathrm{P}\bigl[-X \le q_\alpha(-X)\bigr] - \alpha \ne 0 \;\Rightarrow\; \mathrm{P}\bigl[-X = q_\alpha(-X)\bigr] > 0. \quad (4.11b)$$

Hence, it makes sense to define a $\{0,1\}$-valued random variable $J = J_{X,\alpha}$ with

$$\mathrm{P}[J=1] \;=\; p_\alpha \;=\; 1 - \mathrm{P}[J=0], \quad (4.12a)$$

where

$$p_\alpha \;=\; \begin{cases} \frac{\mathrm{P}[-X \le q_\alpha(-X)] - \alpha}{\mathrm{P}[-X = q_\alpha(-X)]}, & \text{if } \mathrm{P}[-X = q_\alpha(-X)] > 0\,, \\ 0\,, & \text{otherwise.} \end{cases} \quad (4.12b)$$

Proposition 4.7 *Let X, Y be real random variables such that $\mathrm{E}[|Y|] < \infty$ and $\alpha \in (0,1)$ a fixed confidence level. Suppose that there is a random variable J which satisfies (4.12a), (4.12b) and is independent from (X, Y). Define*

$$I \;=\; \mathbf{1}_{\{-X > q_\alpha(-X)\} \cup \{-X = q_\alpha(-X),\, J=1\}}. \quad (4.13a)$$

Then

$$\mathrm{E}[Y \,|\, I = 1] \;=\; (1-\alpha)^{-1} \int_\alpha^1 \mathrm{E}\bigl[Y \,|\, X = -\mathrm{VaR}_u(X)\bigr]\, d\,u. \quad (4.13b)$$

Proof. By (4.12b) and the independence of I and (X, Y) we have

$$\mathrm{P}[I = 1] \;=\; 1 - \alpha. \quad (4.14)$$

It is now straightforward to see that

$$\mathrm{E}[Y\, \mathbf{1}_{\{I=1\}}] = \mathrm{E}[Y\, \mathbf{1}_{\{-X \ge q_\alpha(-X)\}}]$$
$$+\, \mathrm{E}\bigl[Y \,|\, -X = q_\alpha(-X)\bigr]\, \Bigl(\mathrm{P}[-X < q_\alpha(-X)] - \alpha\Bigr). \quad (4.15)$$

Thus, the assertion follows from Corollary 4.6. □

The philosophy behind value-at-risk (VaR) is that the event $\{-X \leq q_\alpha(-X)\}$ is tolerable whereas $\{-X > q_\alpha(-X)\}$ corresponds to a kind of default. Note that

$$\mathrm{P}\big[-X > q_\alpha(-X)\big] \;\leq\; 1-\alpha. \tag{4.16}$$

Hence one might consider I from Proposition 4.7 an indicator of $\{-X > q_\alpha(-X)\}$ modified in a way that enlarges the probability of default. Setting $Y = X$ in (4.13b) shows that ES itself may be regarded as a conditional expectation in a worst case scenario. Replacing X by $Z(u)$ and Y by X_i shows that the same holds for the ES marginal impacts from (4.10).

Observe that (4.13b) is also a statement about how to estimate ES and the ES marginal impacts. Assume that an independent, identically distributed sample $(X_{1,i}, \ldots, X_{d,i})$, $i = 1, \ldots, N$, of the portfolio component returns is given (cf. (4.10)). Let $Z_i = \sum_{j=1}^d u_j\, X_{j,i}$, $i = 1, \ldots, N$.

- First estimate $q_\alpha(-Z)$ from $(Z_1, \ldots, Z_N)$ by some number $\hat{q}$.
- Estimate the probabilities $\mathrm{P}\big[-Z \leq q_\alpha(-Z)\big]$ and $\mathrm{P}[-Z = q_\alpha(-Z)]$. Let p_s and p_e denote the corresponding estimators.
- Determine a sub-sample by taking all those i such that $-Z_i > \hat{q}$ or $-Z_i = \hat{q}$ and an additional independent Bernoulli experiment with success probability $\frac{p_s-\alpha}{p_e}$ (only in case $p_e > 0$) results in 1.
- Estimate $\mathrm{ES}_\alpha(Z)$ and the marginal impacts according to (4.10) as negative averages of this sub-sample.

References

Acerbi, C. (2002) Spectral Measures of Risk: a Coherent Representation of Subjective Risk Aversion. *Journal of Banking & Finance* **26**(7), 1505-1518.

Acerbi, C., Nordio, C., Sirtori, C. (2001) Expected Shortfall as a Tool for Financial Risk Management. *Working paper.* `http://www.gloriamundi.org/var/wps.html`

Acerbi, C., Tasche, D. (2002) On the coherence of Expected Shortfall. *Journal of Banking & Finance* **26**(7), 1487-1503.

Artzner, P., Delbaen, F., Eber, J.-M., Heath, D. (1997) Thinking coherently. *Risk* **10**, no. 11.

Artzner, P., Delbaen, F., Eber, J.-M., Heath, D. (1999) Coherent measures of risk. *Mathematical Finance* **9**(3), 203–228.

Delbaen, F. (1998) Coherent risk measures on general probability spaces. *Working paper, ETH Zürich.* `http://www.math.ethz.ch/~delbaen/`

Denault, M. (2001) Coherent allocation of risk capital. *Journal of Risk* **4**, no. 1, 1–34.

Denneberg, D. (1994) *Non-additive measure and integral.* Kluwer Academic Publishers, Dordrecht.

Embrechts, P., Höing, A., Juri, A. (2001) Using Copulae to bound the Value-at-Risk for functions of dependent risks. *Working paper, ETH Zürich.* `http://www.math.ethz.ch/~baltes/ftp/papers.html`

Embrechts, P., Klüppelberg, C., Mikosch, T. (1997) *Modelling Extremal Events.* Springer-Verlag, Berlin.

Fischer, T. (2001) Coherent risk measures depending on higher moments. *Working paper, Universität Heidelberg.* `http://www.rzuser.uni-heidelberg.de/~tfischer/`

Föllmer, H., Schied, A. (2002) Convex measures of risk and trading constraints. *To appear in: Finance and Stochastics.*

Gouriéroux, C., Laurent, J. P., Scaillet, O. (2000) Sensitivity analysis of values at risk. *Journal of Empirical Finance* **7**, 225–245.

Hallerbach, W. (1999) Decomposing Portfolio Value-at-Risk: a General Analysis. *Discussion paper TI 99-034/2, Tinbergen Institute Rotterdam.* `http://www.tinbergen.nl/`

Koenker, R, Bassett, G. (1978) Regression quantiles. *Econometrica* **46**, 33–50.

Kusuoka, S. (2001) On law invariant coherent risk measures, in: *Advances in Mathematical Economics*, 3, pp. 83–95. *Springer: Tokyo.*

Lemus, G. (1999) *Portfolio Optimization with Quantile-based Risk Measures.* PhD thesis, Sloan School of Management, MIT. `http://citeseer.nj.nec.com/lemus99portfolio.html`

Litterman, R. (1996) Hot Spots$^{\mathrm{TM}}$ and Hedges. *The Journal of Portfolio Management* **22**, 52–75. Special issue.

Luciano, E., Marena, M. (2001) *Value at Risk bounds for portfolios of non-normal returns.* Forthcoming in: Zopoudinis, C. (Editor), New Trends in Banking Management. *Physica.*

PFLUG, G. (2000) *Some remarks on the value-at-risk and the conditional value-at-risk,* in: Uryasev, S. (Ed.), Probabilistic Constrained Optimization: Methodology and Applications. *Kluwer Academic Publishers.* http://www.gloriamundi.org/var/pub.html

ROCKAFELLAR, R.T., URYASEV, S. (2001) Conditional value-at-risk for general loss distributions. *Journal of Banking & Finance* **26**(7), 1443-1471.

SCHMEIDLER, D. (1986) Integral representation without additivity. *Proc. Amer. Math. Soc.* **97**, 255–261.

TASCHE, D. (1999) Risk contributions and performance measurement. *Working paper, Technische Universität München.* http://citeseer.nj.nec.com/tasche99risk.html

TASCHE, D. (2000) Conditional expectation as quantile derivative. *Working paper, Technische Universität München.* http://citeseer.nj.nec.com/tasche00conditional.html

WANG, S. (1996) Premium calculation by transforming the layer premium density. *ASTIN Bulletin* **26**, 71–92.